CONTROL OF SOURCES IN ONE PROBLEM OF HEAT AND MASS TRANSFER IN GAS MIXTURES

N. P. Volkov

UDC 536.2

A necessary condition of controllability for processes of heat and mass transfer in binary gas mixtures and liquid solutions is presented. The necessary and sufficient conditions for invertibility of the system under consideration are obtained.

The problem of the controllability and invertibility of a system with distributed parameters is now an urgent one. For example, [1-3] are devoted to its solution. A similar problem arises in the theory of inverse problems of mathematical physics [4-7], when the question of the existence of solutions of these problems is investigated.

The proper control of processes of chemical technology, in the occurrence of which heat and mass transfer plays the dominant role, is a decisive factor in obtaining output of the highest quality [8]. One of the problems of the control of a thermodiffusion process taking place in a two-component gas mixture is investigated in the present paper.

Let a binary gas mixture fill a convex confined region $\mathscr{D}$ of R^3 with a boundary $\partial\mathscr{D}$ of class C^1; n is the outward normal to the surface $\partial\mathscr{D}$. Under the assumption that heat and mass transfer take place at a constant pressure, the appearance and disappearance of one or the other component of the gas mixture is due only to phase or chemical transitions, which are not accompanied by heat release or absorption, and that all the thermophysical parameters are constant, we obtain the following system of parabolic equations [9]:

$$\frac{\partial\rho_{10}}{\partial t} = D\Delta\rho_{10} + \frac{k_T D}{T}\Delta\theta + I_1, \quad \frac{\partial\theta}{\partial t} = \frac{DQ^*}{c}\Delta\rho_{10} + \frac{k}{c\rho}\Delta\theta + \frac{(h_2 - h_1)}{c\rho}I_1. \tag{1}$$

The first of these equations expresses the law of conservation of mass while the second expresses the law of conservation of thermal energy.

At the initial time t = 0 the relative concentration ρ_{10} of the first component and the temperature θ of the mixture are given, i.e.,

$$\rho_{10}(x, 0) = \varphi_1(x), \quad \theta(x, 0) = \varphi_2(x). \tag{2}$$

The process under consideration takes place in the closed volume $\mathscr{D}$. Assuming that it is thermally insulated in this case, we obtain the equality to zero of the densities of the diffusional and heat fluxes at the boundary $\partial\mathscr{D}$, i.e.,

$$\left.\frac{\partial\rho_{10}}{\partial n}\right|_{\partial\mathscr{D}\times[0,\,T']} = 0, \quad \left.\frac{\partial\theta}{\partial n}\right|_{\partial\mathscr{D}\times[0,\,T']} = 0. \tag{3}$$

Introducing the notation

$$u(x, t) = \begin{pmatrix} \rho_{10}(x, t) \\ \theta(x, t) \end{pmatrix}, \quad A = \begin{pmatrix} D & \frac{k_T}{T}D \\ \frac{DQ^*}{c} & \frac{k}{c\rho} \end{pmatrix},$$

Moscow Engineering Physics Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 49, No. 6, pp. 936-940, December, 1985. Original article submitted May 17, 1985.

 0022-0841/85/4906-1422$09.50

$$b(x,t)=\begin{pmatrix}1\\ \dfrac{h_2(x,t)-h_1(x,t)}{c\rho}\end{pmatrix},\quad \varphi(x)=\begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix}$$

and assuming that the quantity k_T/T is a constant, since it varies insignificantly during the entire process, and $k/\rho > DQ^*k_T/T$, we obtain the following mixed problem for a uniformly parabolic system in vector form:

$$\frac{\partial u}{\partial t}(x,t)=A\Delta u(x,t)+b(x,t)I_1(x,t),\quad (x,t)\in\Omega=\mathscr{D}\times(0,T'), \tag{4}$$

$$u(x,0)=\varphi(x),\quad x\in\mathscr{D}, \tag{5}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma}=0,\quad (x,t)\in\Gamma=\partial\mathscr{D}\times[0,T']. \tag{6}$$

The control problem consists in the determination of the function (the control function) $I_1(x,t)\in H^{\alpha,\alpha/2}(\bar\Omega)$ and the solution $u(x,t)\in H^{2+\alpha,1+\alpha/2}(\bar\Omega)$ corresponding to it (the vector function of state) for the problem (4)-(6) such that the condition

$$u(x,t_1)=\psi(x),\quad x\in\mathscr{D},\quad 0<t_1<T', \tag{7}$$

is satisfied, i.e., at the fixed time t_1 the relative concentration of the first component and the temperature of the mixture must coincide with the values assigned in advance.

Let us find the necessary condition for controllability of the transition of the system (4) from the initial state $\varphi(x)$ to the final state $\psi(x)$ in the time t_1.

Let a solution of the problem (4)-(7) exist; then from the results of [10], when the consistency conditions are satisfied, it follows that the solution of the direct problem (4)-(6) can be represented in the form

$$u(x,t)=\int_{\mathscr{D}}G(x,y,t,0)\varphi(y)\,dy+\int_0^t\int_{\mathscr{D}}G(x,y,t,\tau)b(y,\tau)I_1(y,\tau)\,dyd\tau, \tag{8}$$

where $G(x, y; t, \tau)$ is Green's function for the problem under consideration.

Since the condition (7) must be satisfied at $t = t_1$, we have the equality

$$\psi(x)=\int_{\mathscr{D}}G(x,y,t_1,0)\varphi(y)\,dy+\int_0^{t_1}\int_{\mathscr{D}}G(x,y,t_1,\tau)b(y,\tau)I_1(y,\tau)\,dyd\tau.$$

The following the procedure expounded in [6], we can show that the condition

$$\operatorname{rank}[P(x);\ w(x)]=\operatorname{rank}[P(x)]\quad\text{for all}\quad x\in\mathscr{D}, \tag{9}$$

where

$$w(x)=\psi(x)-\int_{\mathscr{D}}G(x,y,t_1,0)\varphi(y)\,dy;\quad P(x)=\int_0^{t_1}\int_{\mathscr{D}}G(x,y,t_1,\tau)b(y,\tau)\times b^*(y,\tau)G^*(x,y,t_1,\tau)\exp[-(t_1-\tau)^{-2}]\,dyd\tau$$

(* denotes transposition), is necessary for the controllability of the system (4).

Another approach to finding $u(x, t)$ and $I_1(x, t)$ from the given classes is based on the method of solving the inverse problem (4)-(6) with the condition of overdetermination of the type

$$u(x_0,t)=\mu(t),\quad t\in[0,T'],\quad x_0\in\mathscr{D}. \tag{10}$$

By analogy with the preceding problem, we designate

$$\tilde w(t)=\mu(t)-\int_{\mathscr{D}}G(x_0,y,t,0)\varphi(y)\,dy,$$

$$\tilde P(t)=\int_0^t\int_{\mathscr{D}}G(x_0,y,t,\tau)b(y,\tau)b^*(y,\tau)G^*(x_0,y,t,\tau)\exp[-(t-\tau)^{-2}]\,dyd\tau.$$

Then, when the correspondence

$$\varphi(x_0) = \mu(0) \tag{11}$$

is satisfied, the condition

$$\text{rank}\,[\tilde{P}(t);\ \ \tilde{w}(t)] = \text{rank}\,[\tilde{P}(t)] \quad \text{for any} \qquad t \in [0,\ T'] \tag{12}$$

is necessary and sufficient for the existence of the pair $\{u(x,\ t),\ I_1(x,\ t)\} : u \in H^{2+\alpha,\ 1+\alpha/2}(\bar{\Omega})$, $I_1 \in H^{\alpha,\alpha/2}(\bar{\Omega})$, yielding the solution of the problem (4)-(6), (10).

Before proceeding to the proof of this statement, we consider the auxiliary problem (4)-(6) with an overdetermination condition of the type

$$u(x_0,\ t_1) = \mu_1 \equiv \mu(t_1), \tag{13}$$

where $(x_0,\ t_1)$ is a fixed point of $\mathscr{D} \times (0,\ T']$.

Using the results of [6], one can easily show that a solution $\{u(x,\ t),\ I_1(x,\ t)\}$ of the inverse problem (4)-(6), (13) exists in the indicated classes when and only when $\text{rank}[\tilde{P}(t_1);\ \tilde{w}(t_1)] = \text{rank}[\tilde{P}(t_1)]$.

We prove that the condition (11) is necessary from the contrary, i.e., suppose a pair $\{u(x,\ t),\ I_1(x,\ t)\}$ exists for which the conditions (4)-(6), (10), (11) are satisfied, while there exists a $t_1 \in [0,\ T']$ such that $[\tilde{P}*t_1);\ \tilde{w}(t_1)] > \text{rank}[\tilde{P}(t_1)]$, but then the inverse problem (4)-(6), (13) has no solution, on the strength of the criterion persented above. Consequently, the problem (4)-(6), (10) also cannot have a solution from the indicated classes. A contradiction is obtained.

Now let the condition (12) be satisfied. This means that a $z \in R^2$ exists such that

$$\tilde{w}(t) = \tilde{P}(t)\,z \quad \text{for all} \qquad t \in [0,\ T']. \tag{14}$$

As was shown in [10], in the classes under consideration the problem (4)-(6) is equivalent to the system of internal equations (8), or for $x = x_0$ we obtain the system

$$\int_0^t \int_{\mathscr{D}} G(x_0,\ y,\ t,\ \tau)\, b(y,\ \tau)\, I_1(y,\ \tau)\, dy d\tau = \tilde{w}(t).$$

We seek the function $I_1(x,\ t)$ in the form $I_1(x,\ t) = g(x)f(t)$, where $f(t)$ is unknown while $g(x)$ is chosen so that the kernel $K(t,\ \tau) = \int_{\mathscr{D}} G(x_0,\ y,\ t,\ \tau) \times b(y,\ \tau)\, g(y)\, dy$ satisfies the conditions required in [5]. Then for the determination of $f(t)$ we have a Volterra system of integrals equations of the first kind,

$$\int_0^t K(t,\ \tau)\, f(\tau)\, d\tau = \tilde{w}(t),$$

which, by virtue of the condition (14), is consistent and can be reduced to a Volterra system of equations of the second kind, solvable in the class $H^{\alpha/2}([0,T'])$. Then for $f(t)$ we determine $I_1(x,\ t)$, while from Eq. (8) we determine $u(x,\ t)$, i.e., the solution of the inverse problem (4)-(6), (10) is found.

In the problem under consideration the condition (12) expresses the criterion of invertibility (to the left) [3] of the system (4).

From the proof it is seen that the solution of the inverse problem (4)-(6), (10) is not unique. To obtain a unique solution one must set an additional restriction on the state, such as an integral one. And then one must solve the problem of optimum control.

In conclusion, we note that similar problems also arise for processes of heat and mass transfer in liquid solutions.

NOTATION

$\rho_{10}(x, t)$, relative concentration of the first component in the region $\Omega = \{(x, t) \in \mathcal{D} \times (0, T'), \mathcal{D} \subset R^3\}$; $\theta(x, t)$, temperature of the mixture in the region Ω on the Celsium scale; T, temperature of the mixture on the Kelvin scale; k_T, thermodiffusion coefficient; D, coefficient of interdiffusion of the mixture; I_1, density of internal sources of the first component of the mixture; Q*, specific heat of isothermal transfer; c, specific heat of the mixture; ρ, density of the mixture; h_i, specific enthalpy of the i-th component; Δ, Laplace operator in R^3; k, coefficient of thermal conductivity of the mixture.

LITERATURE CITED

1. H. O. Fattorini and D. L. Russel, "Exact controllability theorems for linear parabolic equations in one space dimension," Arch. Ration. Mech. Anal., 43, No. 4, 272-292 (1971).
2. J. L. Lions, Optimum Control of Systems Governed by Partial Differential Equations, Springer-Verlag, Berlin—New York (1971).
3. V. T. Borukhov, "Inversion of linear, time-invariant, dynamic systems with distributed parameters," Avtom. Telemekh., No. 5, 29-36 (1982).
4. A. I. Prilepko, "Inverse problems of potential theory (elliptic, parabolic, and hyperbolic equations and transfer equations)," Mat. Zametki, 14, No. 5, 755-767 (1973).
5. A. E. Uzlov, "Inverse problems of determination of coefficients and the right side for equations and systems of the parabolic type," Dissertation for Degree in Physicomathematical Sciences, Moscow (1983).
6. N. P. Volkov, "Existence of solutions of certain inverse problems of determination of the right side of a parabolic system," in: Functional Methods in Problems of Mathematical Physics [in Russian], Energoatomizdat, Moscow (1984), pp. 8-11.
7. V. P. Mishin and O. M. Alifanov, "Inverse problems of heat exchange: regions of application in the design and tests of technical equipment," Inzh.-Fiz. Zh., 42, No. 2, 181-192 (1982).
8. Ya. Didushinskii, Principles of the Design of Catalytic Reactors [in Russian], Khimiya, Moscow (1972).
9. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).
10. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc. (1968).